

# Defect formation from defect–anti-defect annihilations

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## Abstract

We show that when a topological defect with extended world-volume annihilates with an anti-defect, there arise topological defects with dimensions less than those of the original defects by one. Domain wall annihilations create vortices while monopole-string annihilations result in instantons. We find that twisted domain wall rings are vortices, whereas twisted monopole rings are instantons.

## I. INTRODUCTION

When a particle and its anti-particle meet, they annihilate and turn to energy or decay into other particles. For instance, an electron and a positron on collision annihilate each other to create a photon. Similarly, when a soliton and an anti-soliton collide, they also annihilate and usually decay into elementary excitations. For instance, in superfluids, a vortex and an anti-vortex pair annihilate on collision and decay into phonons. A question that arises along similar line is what happens if an extended object and its counter object collide. Some years ago, this was studied in string theory for a Dirichlet(D)  $p$ -brane, which is an extended object (soliton) with a world-volume with a  $p$ -dimension. It has been found that the annihilation of a  $Dp$ -brane with an anti- $Dp$ -brane results in the creation of  $D(p-2)$ -branes [1]. Then, the question that arises is whether such a process exists for solitons as extended objects in field theory.

In this paper, we show that when a soliton and an anti-soliton collide and annihilate, there arise topological defects with dimensions less than those of the original defects by one. This is inevitable when topological defects have a  $U(1)$  modulus (collective coordinate). The examples are 1) when a domain wall and an anti-domain wall annihilate in  $2+1$  dimensions, there appear vortices, while 2) when a monopole string and anti-monopole string annihilate in  $4+1$  dimensions, there appear Yang-Mills instantons. As a byproduct of our results, we find that twisted domain wall (monopole) rings are equivalent to vortices (instantons).

## II. VORTICES FROM WALL-ANTI-WALL ANNIHILATION

As the simplest example, we first consider a domain wall pair annihilation. Domain walls are extended objects in  $d = 2 + 1$  dimensions or more. We consider the  $CP^1$  model, also known as the  $O(3)$  model, which describes ferromagnets. In order to admit domain walls, we consider the potential term admitting two minima, which gives a one-axis isotropy in the context of the magnetism. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} - m^2(1 - n_3^2), \quad \mathbf{n}^2 = 1. \quad (1)$$

with a three-vector of scalar fields,  $\mathbf{n} = (n_1(x), n_2(x), n_3(x))$ . This model has two discrete vacua,  $n_z = \pm 1$ , corresponding to the north and south poles of the sphere. It is also known as the massive  $CP^1$  model [2], which shows supersymmetry when fermions are suitable

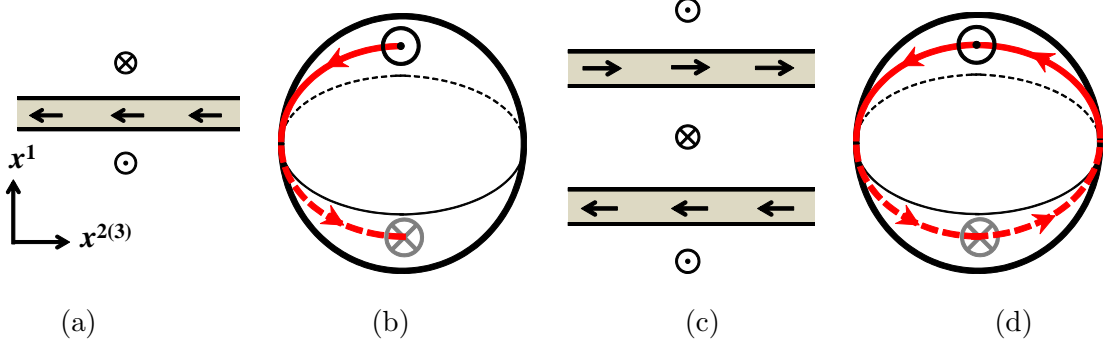


FIG. 1: (a,b) Single domain wall and (c,d) a pair of a domain wall and an anti-domain wall in the  $\mathbf{CP}^1$  model. (a) A single domain wall. The wall is perpendicular to the  $x^1$ -axis. The arrows denote points in the  $\mathbf{CP}^1$ . (b) The  $\mathbf{CP}^1$  target space. The north and south poles are denoted by  $\odot$  and  $\otimes$ , respectively. The path connecting them represents the map from the path in (a) along the  $x^1$ -axis in the real space from  $x^1 \rightarrow -\infty$  to  $x^1 \rightarrow +\infty$ . The path in the  $\mathbf{CP}^1$  target space passes through one point on the equator, which is represented by “ $\leftarrow$ ” in (a) in this example. In general, the  $U(1)$  zero modes are localized on the wall.

(c) The wall and anti-wall configuration. The  $U(1)$  zero modes on the wall and anti-wall are opposite to each other. (d) The path represents the map from the wall and anti-wall configuration.

introduced in it, although supersymmetry is not essential in our study. It is convenient to use the  $\mathbf{CP}^1$  projective coordinate  $u$ , considering which the Lagrangian can be rewritten as [2]

$$\mathcal{L} = \frac{\partial_\mu u^* \partial^\mu u - m^2 |u|^2}{(1 + |u|^2)^2} \quad (2)$$

with  $\mathbf{n} = \Phi^\dagger \sigma \Phi$ , where  $\Phi^T = (1, u)/\sqrt{1 + |u|^2}$ . The two vacua,  $n_3 = +1$  and  $n_3 = -1$ , are mapped to  $u = 0$  and  $u = \infty$ , respectively.

There exists a domain wall interpolating the two discrete vacua [2]

$$u_{dw} = e^{m(x^1 - x_0^1) + i\varphi} \quad (3)$$

with width  $1/m$  and tension  $m$ . Here,  $x_0^1$  and  $\varphi$  are real constants called moduli parameters or collective coordinates. They are Nambu-Goldstone modes associated with broken translational and internal  $U(1)$  symmetries, respectively. A domain wall is mapped to a large circle starting from the north pole, denoted by  $\odot$ , and ending up at the south pole, denoted

by  $\otimes$ , in the  $\mathbf{CP}^1$  target space (Fig. 1 (b)). An anti-domain wall is obtained simply by introducing a minus sign in front of  $x^1 - x_0^1$  in Eq. (3).

Next, we consider a domain wall and an anti-domain wall. Here, the  $U(1)$  zero modes of the wall and anti-wall are considered to be opposite, as in Fig. 1 (c). The configuration is mapped to a loop in the  $\mathbf{CP}^1$  target space (Fig. 1 (d)). However, this configuration is unstable as it should end up with the vacuum with the up-spin  $\odot$ . In the decaying process, the loop is unwound from the south pole in the target space. The unwinding of the loop can be achieved in two topologically inequivalent ways, which are schematically shown in Fig. 2 (e) and (f). In the real space, at first, a bridge connecting the two walls is created as in Fig. 2 (a) and (d). Here, there exist two possibilities of the spin structure of the bridge, corresponding to two ways of the unwinding processes. Along the bridge in the  $x^1$ -direction, the spin rotates (a) anti-clockwise or (b) clockwise on the equator of the  $\mathbf{CP}^1$  target space. Let us label these two kinds of bridges by “ $\downarrow$ ” and “ $\uparrow$ ”, respectively.

In the next step, the bridge is broken into two pieces, as in Fig. 2 (a) and (d), with the vacuum state, i.e., the up-spin  $\odot$  state, filled between them. Let us again label these two kinds of holes by “ $\downarrow$ ” and “ $\uparrow$ ”, respectively. In either case, the two regions separated by the domain walls are connected through a hole created by the decay of the domain walls. Once created, these holes grow, thus reducing the wall tension.

Several holes are created during the entire decaying process. Let us focus on a pair of two neighboring holes. One can find a ring of a domain wall between the holes, as shown in Fig. 3. Here, since there exist two kinds of holes ( $\uparrow$  and  $\downarrow$ ), there exist four possibilities of the rings, (a)  $\uparrow\downarrow$ , (b)  $\downarrow\uparrow$ , (c)  $\uparrow\uparrow$ , and (d)  $\downarrow\downarrow$  (Fig. 3). Clearly, the rings of types (c) or (d) can decay and end up with the vacuum state  $\odot$ . However, the decay of the rings of types (a) and (b) is topologically forbidden, because of a nontrivial winding of the spin along the rings.

What are these topologically protected rings of a domain wall? They are nothing but lumps or sigma model instantons [3], which are also known as 2D skyrmions in condensed matter. The solutions can be written as ( $z \equiv x^1 + ix^2$ )

$$u = u_0 = \lambda/(z - z_0) \quad \text{or} \quad u = \bar{u}_0, \quad (4)$$

as a lump or an anti-lump, where  $z_0 \in \mathbf{C}$  represent the positions of the lump and  $\lambda \in \mathbf{C}^*$ , with  $|\lambda|$  and  $\arg \lambda$  representing the size and  $U(1)$  orientation of the lump, respectively. In

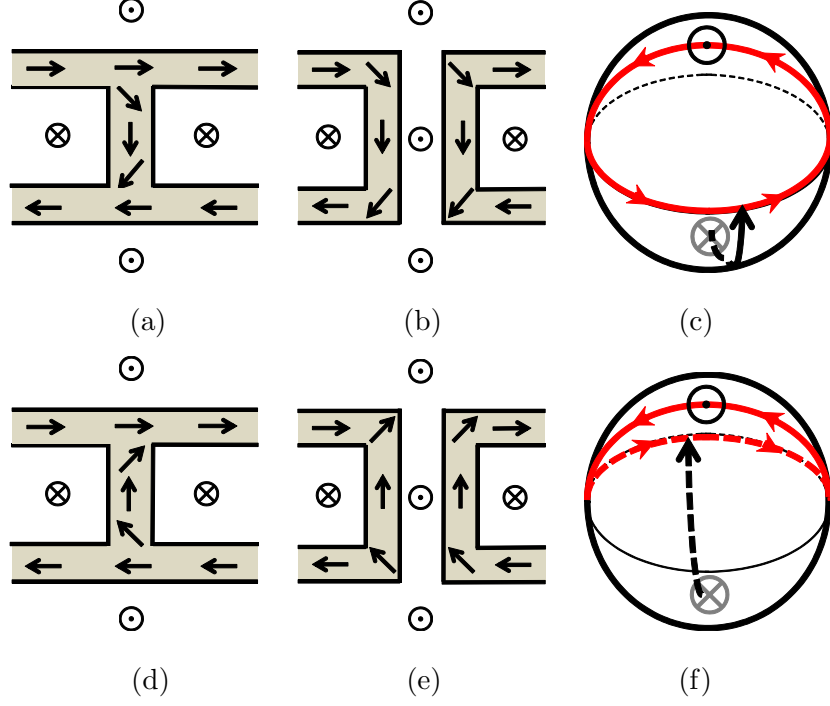


FIG. 2: Decaying processes of the wall and anti-wall. (a,d) A bridge is created between the wall and the anti-wall. In this process, there are two possibilities of the  $\mathbf{CP}^1$  structure along the bridge. (b,e) The upper and lower regions are connected by breaking the bridge. (c,f) Accordingly, the loop in the  $\mathbf{CP}^1$  target space is unwound in two ways.

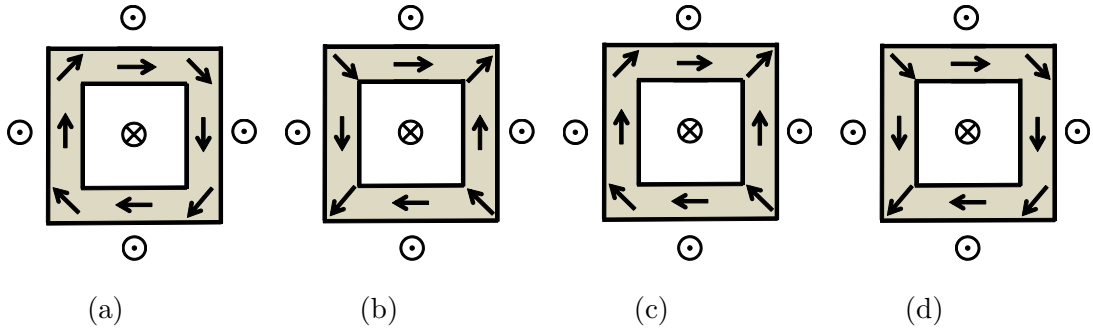


FIG. 3: Stable and unstable rings. (a,b) Stable rings. The phase winds once (the winding number is  $\pm 1$ ) along the rings. Total configurations are lumps with a non-trivial element,  $\pm 1$ , of the second homotopy group,  $\pi_2$ . (c,d) Unstable rings. The phase does not wind (the winding number is 0) along the rings. They decay into ground state (up pseudo-spin).

fact, these configurations can be shown to have a nontrivial winding in the second homotopy group,  $\pi_2(\mathbf{CP}^1) \simeq \mathbf{Z}$ , which can be calculated from  $\frac{1}{2\pi} \int d^2x \frac{i(\partial_x u^* \partial_y u - \partial_y u^* \partial_x u)}{(1 + |u|^2)^2}$ . (a) and (b) belong to, respectively, +1 and -1 of  $\pi_2(\mathbf{CP}^1)$ . That is, they are a lump and an anti-lump, respectively.

In the context of magnetism, these configurations are referred to as “bubble domains.”

In the presence of the potential term, this solution is actually unstable to shrink. It can be found that the size  $|\lambda|$  tends to zero when the configuration in Eq. (4) is substituted in the energy term corresponding to the Lagrangian (2). In order to avoid this, one can consider the  $\mathbf{CP}^1$  model as a low-energy theory for the  $U(1)$  gauge theory coupled with two complex scalar fields, as follows:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{e^2} (\partial_\mu \Sigma)^2 + |D_\mu \Phi|^2 - V, \\ V &= \frac{e^2}{2} (\Phi^\dagger \Phi - v^2)^2 + \Phi^\dagger (\Sigma \mathbf{1}_2 - M)^2 \Phi\end{aligned}\tag{5}$$

with complex scalar fields  $\Phi = (\phi^1, \phi^2)^T$  and a real scalar field  $\Sigma$ ; here,  $M = \text{diag.}(m_1, m_2)$  with  $m_1 > m_2$  and  $m_1 - m_2 = m$ . This Lagrangian is the bosonic part of the  $\mathcal{N} = 2$  supersymmetric gauge theories, where  $v^2$  is called the Fayet-Illiopoulos parameter. However, supersymmetry is not essential in our study.

Lumps are replaced with so-called semi-local vortices [4], which also have size moduli in the absence of mass  $m$ . In the presence of the mass, however, they shrink to the minimum size vortices, *i.e.*, local (Abrikosov-Nielsen-Olesen) vortices [5].

In  $d = 3 + 1$  dimensions, domain walls have two spatial dimensions in their world-volume. When a domain-wall pair decay occurs, there appear two-dimensional holes, which are labeled as  $\downarrow$  or  $\uparrow$  in Fig. 2 (b) or (e), respectively. Along the boundary of these two kinds of holes, there appear vortex strings, which generally create vortex rings. This process can be numerically validated [6]. The vortex rings decay into fundamental excitations in the end.

### III. INSTANTONS FROM MONOPOLE-ANTI-MONOPOLE ANNIHILATION

As the second example, let us consider a pair annihilation of 't Hooft-Polyakov monopoles. 't Hooft-Polyakov monopoles are point-like topological defects in  $SU(2)$  Yang-Mills theory, coupled with Higgs scalar fields in the triplet representation in  $d = 3 + 1$  dimensions [7]. In

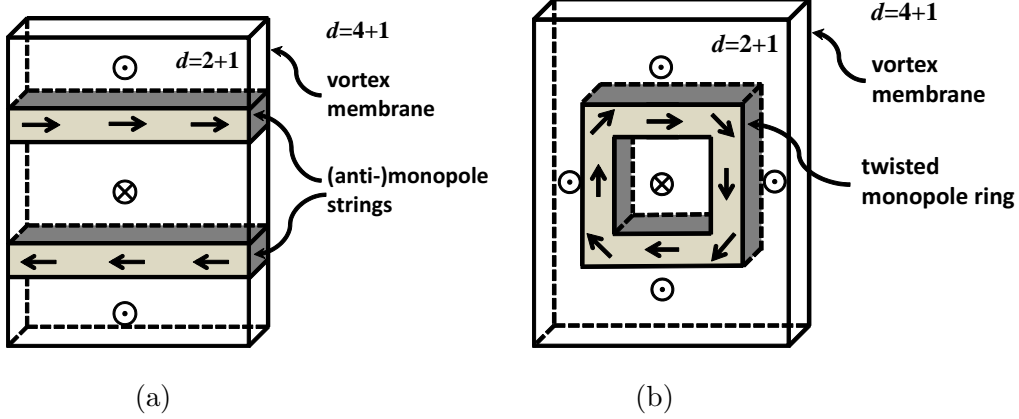


FIG. 4: (a) A pair of a monopole string and an anti-monopole-string inside a non-Abelian vortex. (b) A twisted monopole ring inside a non-Abelian vortex.

order to create lower-dimensional defects, the topological defects to be annihilated require at least one world-volume direction. Therefore, we consider monopole strings in  $d = 4 + 1$  dimensions as the minimum set-up.

The approach involves putting the system into the Higgs phase by considering the  $U(2)$  gauge theory instead of  $SU(2)$ , where magnetic fluxes from a monopole are squeezed into vortices [8]. The Lagrangian in the  $d = 4 + 1$  dimensions, which we consider, is given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2g^2} \text{tr} (D_\mu \Sigma)^2 + \text{tr} D_\mu H^\dagger D^\mu H - V, \\ V &= g^2 \text{tr} (HH^\dagger - v^2 \mathbf{1}_2)^2 + \text{tr} [H(\Sigma \mathbf{1}_2 - M)^2 H^\dagger], \end{aligned} \quad (6)$$

with two complex scalar fields in the fundamental representation of  $SU(2)$ , summarized as a two by two complex matrix  $H$ , and with the real matrix scalar fields  $\Sigma$  in the adjoint representation. The mass matrix is given as  $M = \text{diag.}(m_1, m_2)$ , with  $m_1 > m_2$  and  $m_1 - m_2 = m$ . Again, this Lagrangian can be made  $\mathcal{N} = 2$  supersymmetric by suitably adding fermions; however, supersymmetry is not essential in our study.

In the massless limit,  $m = 0$ , this model admits a non-Abelian  $U(2)$  vortex solution  $H = \text{diag.}(f(r)e^{i\theta}, v)$  [9]. The solution breaks the vacuum symmetry  $SU(2)_{\text{C+F}}$  into  $U(1)$ , and there appear  $\mathbf{CP}^1 \simeq U(2)_{\text{C+F}}/U(1)$  Nambu-Goldstone modes. The low-energy effective theory on the  $2 + 1$ -dimensional vortex world-volume is the  $\mathbf{CP}^1$  model. In the presence of mass, *i.e.*  $m \neq 0$ , the  $SU(2)_{\text{C+F}}$  symmetry is explicitly broken, inducing the same type of

mass matrix in the  $d = 2 + 1$  vortex effective theory;

$$\mathcal{L}_{\text{vort.eff.}} = 2\pi v^2 |z_0|^2 + \frac{4\pi}{g^2} \left[ \frac{\partial_\mu u^* \partial^\mu u - m^2 |u|^2}{(1 + |u|^2)^2} \right]. \quad (7)$$

Let us construct a domain wall in this effective theory. The domain wall tension is  $4\pi m/g^2$ , which coincides with the monopole mass. It is equivalent to the monopole charge because the monopole is a BPS state. Therefore, the wall in the vortex theory is nothing but a monopole string from the bulk point of view [8].

We then consider a pair of a monopole string and an anti-monopole string in the vortex world-volume, as in Fig. 4 (a). The result in the last section immediately implies that after a monopole pair annihilation, we obtain lumps [3] in the vortex world-volume, as shown in Fig. 4 (b). As shown in [10], lumps inside a non-Abelian vortex are nothing but Yang-Mills instantons [11], which are point-like topological objects from the  $d = 4 + 1$  bulk point of view. This can also be inferred from the lump energy, which coincides with the instanton energy.

Finally, we send the FI parameter  $v^2$  to zero with the system going back to the Coulomb phase. Then, without the help of non-Abelian vortices, the monopole-string pair annihilates, resulting in the creation of (anti-)instantons.[14] We also have found that a twisted monopole ring is an instanton.

#### IV. SUMMARY AND DISCUSSION

In conclusion, when a pair of a defect and an anti-defect with opposite  $U(1)$  moduli annihilates, there appear (anti-)defects with dimensions less than those of the original defects by one. We have demonstrated this phenomenon in two examples: 1) when a domain wall and an anti-domain wall annihilate in 2+1 dimensions, there appear (anti-)vortices, and 2) when a monopole string and an anti-monopole string annihilate in 4+1 dimensions, there appear (anti-)Yang-Mills instantons. We have found that twisted domain wall rings and twisted monopole rings are vortices and instantons, respectively.

Another example is that of superconducting cosmic strings that have a  $U(1)$  zero mode [12]. When a pair of superconducting strings annihilates, with opposite  $U(1)$  moduli, there can appear a vorton, *i.e.* a twisted vortex ring. It is an open question if this vorton can be identified with a monopole in some setting.



Studies on defect-anti-defect annihilations in this paper have been restricted to semi-classical analysis. Quantum mechanical decay of metastable topological defects was studied before [13]. The extension to quantum mechanical decay of defect-anti-defect annihilation remains as an important future work.

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[14] In the Coulomb phase, monopole strings can move in their three codimensions, but in the Higgs phase they can move in only one codimension inside a vortex. Therefore, monopole strings generally collide at an angle. In this case, they reconnect with each other, resulting in (un)twisted monopole rings, as discussed in this paper. We can show that such a reconnection always occurs owing to the fact that when two monopoles collide head on in  $d=3+1$  dimensions, they scatter at a  $90^\circ$  angle.